REPRESENTATIONS OF MODULAR SKEW GROUP ALGEBRAS

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Abstract. In this paper we study representations of skew group algebras ΛG, where Λ is a connected, basic, finite-dimensional algebra (or a locally finite graded algebra) over an algebraically closed field k with characteristic $p \geqslant 0$, and G is an arbitrary finite group each element of which acts as an algebra automorphism on Λ. We characterize skew group algebras with finite global dimension or finite representation type, and classify the representation types of transporter categories for $p \neq 2, 3$. When Λ is a locally finite graded algebra and the action of G on Λ preserves grading, we show that ΛG is a generalized Koszul algebra if and only if so is Λ.

1. Introduction

Let Λ be a a connected, basic, finite-dimensional algebra over an algebraically closed filed k with characteristic $p \geqslant 0$, and G is an arbitrary finite group each element of which acts as an algebra automorphism on Λ. The action of G on Λ determines a finite-dimensional algebra ΛG, called *skew group algebra* of Λ. Skew group algebras include ordinary group algebras as special examples when the action of G on Λ is trivial.

Representations and homological properties of skew group algebras (or more generally, group-graded algebras and crossed products) have been widely studied, see [4, 6, 15, 18, 19, 20, 26]. When $|G|$, the order of G, is invertible in k, it has been shown that ΛG and Λ share many common properties. For example, ΛG is of finite representation type (self-injective, of finite global dimension, an Auslander algebra) if and only so is Λ ([19]); if Λ is piecewise hereditary, so is ΛG ([7]). If Λ is a positively graded algebra and the action of G preserves grading, then ΛG is a Koszul algebra if and only if so is Λ, and the Yoneda algebra of ΛG is also a skew group algebra of the Yoneda algebra of Λ ([16]). However, when G is a modular group, many of the above results fail. Therefore, it makes sense to ask under what conditions ΛG and Λ still share these properties.

As for ordinary group algebras, it is natural to use the induction-restriction procedure and consider the relatively projective property of a ΛG-module M with respect to the skew group algebra ΛS, where S is a Sylow p-subgroup of G. This theory has been described in [4, 15]. Using some elementary arguments, we show

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that ΛG is of finite representation type or of finite global dimension if and only if so is ΛS. Therefore, we turn to study the representations of ΛS.

Suppose that Λ has a set of primitive orthogonal idempotents $E = \{e_i\}_{i=1}^n$ such that $\sum_{i=1}^n e_i = 1$ and E is closed under the action of S. This condition holds in many cases, for example, when G acts on a finite quiver, a finite poset, or a finite category, etc, and Λ is the corresponding path algebra, incidence algebra, or category algebra. Under this assumption, we show that if ΛS is of finite representation type (for $p \geqslant 5$) or of finite global dimension, then the action of S on E must be free. In this situation, ΛS is the matrix algebra over Λ^S, the subalgebra of Λ constituted of all elements in Λ fixed by the action of S. Therefore, ΛS is Morita equivalent to Λ^S, and we can prove:

Theorem 1.1. *Let Λ be a basic finite-dimensional k-algebra and G be a finite group acting on Λ. Let $S \leqslant G$ be a Sylow p-subgroup. Suppose that there is a set of primitive orthogonal idempotents $E = \{e_i\}_{i=1}^n$ such that E is closed under the action of S and $\sum_{i=1}^n e_i = 1$. Then:*

(1) $\operatorname{gldim} \Lambda G < \infty$ *if and only if* $\operatorname{gldim} \Lambda < \infty$ *and S acts freely on E. Moreover, if* $\operatorname{gldim} \Lambda G < \infty$*, then* $\operatorname{gldim} \Lambda G = \operatorname{gldim} \Lambda$.
(2) *ΛG is an Auslander algebra if and only if so is Λ and S acts freely on E.*
(3) *Suppose that $p \neq 2, 3$ and Λ is not a local algebra. Then ΛG is of finite representation type if and only if S acts freely on E, and Λ is of finite representation type.*

Let $\mathcal{P}$ be a finite connected poset on which every element in G acts as an automorphism. The Grothendick construction $\mathcal{T} = G \propto \mathcal{P}$ is called a *transporter category*. It is a *finite EI category*, i.e., every endomorphism in $\mathcal{T}$ is an isomorphism. Representations of transporter categories and finite EI categories have been studied in [9, 10, 21, 22, 24, 25]. In a paper [25], Xu showed that the category algebra $k\mathcal{T}$ is a skew group algebra of the incidence algebra $k\mathcal{P}$. We study its representation type for $p \neq 2, 3$, and it turns out that we can only get finite representation type for very few cases:

Theorem 1.2. *Let G be a finite group acting on a connected finite poset $\mathcal{P}$ and suppose that $p \neq 2, 3$. Then the transporter category $\mathcal{T} = G \propto \mathcal{P}$ is of finite representation type if and only if one of the following conditions is true:*

(1) *$|G|$ is invertible in k and $\mathcal{P}$ is of finite representation type;*
(2) *$\mathcal{P}$ has only one element and G is of finite representation type.*

When Λ is a locally finite graded algebra and the action of G preserves grading, the skew group algebra ΛG has a natural grading. Therefore, it is reasonable to characterize its Koszul property. Since $|G|$ might not be invertible in k, the degree 0 part of ΛG might not be semisimple, and the classical Koszul theory cannot apply. Thus we use a generalized Koszul theory developed in [11, 12], which does not demand the semisimple property. Using the generalized definition of Koszul algebras, we prove that ΛG is a generalized Koszul algebra if and only if so is Λ. Moreover, a careful check tells us that the argument in Martinez's paper [16] works as well in the general situation, so we establish the following result.

Theorem 1.3. *Let M be a graded ΛG-module and let $\Gamma = \operatorname{Ext}^*_\Lambda(M, M)$. Then:*

(1) *The ΛG-module $M \otimes_k kG$ is a generalized Koszul ΛG-module if and only if as a Λ-module M is generalized Koszul. In particular, ΛG is a generalized*

Koszul algebra over $(\Lambda G)_0 = \Lambda_0 \otimes_k kG$ if and only if Λ is a generalized Koszul algebra over Λ_0.

(2) *If as a Λ-module M is generalized Koszul, then $\operatorname{Ext}^*_{\Lambda G}(M \otimes_k kG, M \otimes_k kG)$ is isomorphic to the skew group algebra ΓG as graded algebras.*

This paper is organized as follows. In Section 2 we describe the induction-restriction procedure and use it to show that ΛG and ΛS share some common properties. In Section 3 we study representations of the skew group algebra ΛS and prove the first theorem. Transporter categories and its representation type are considered in Section 4, where we prove the second theorem. In the last section we briefly describe a generalized Koszul theory and apply it to characterize the Koszul property of ΛG when Λ is graded and the action of G preserves grading.

We introduce the notation and convention here. For $\lambda \in \Lambda$ and $g \in G$, we denote the image of λ under the action of g by ${}^g\lambda$ or $g(\lambda)$. The skew group algebra ΛG as a vector space is $\Lambda \otimes_k kG$, and the multiplication is defined by $(\lambda \otimes g) \cdot (\mu \otimes h) = \lambda({}^g\mu) \otimes gh$. To simplify the notation, we denote $\lambda \otimes g$ by λg. Correspondingly, the multiplication is $(\lambda g) \cdot (\mu h) = \lambda({}^g\mu)gh$ or $\lambda g(\mu)gh$. The identities of Λ and G are denoted by 1 and 1_G respectively.

All modules we consider in this paper are left finitely generated modules (or left locally finite graded modules when Λ is a locally finite graded algebra). Composite of maps, morphisms and actions is from right to left. To simplify the expression of statements, we view the zero module as a projective or a free module. When $p = 0$, the Sylow p-group is the trivial group.

2. Induction and Restriction

Let $H \leqslant G$ be a subgroup of G. Then ΛH is a subalgebra of ΛG. As for group algebras, we define induction and restriction. Explicitly, for a ΛH module V, the induced module is $V \uparrow_H^G = \Lambda G \otimes_{\Lambda H} V$, where ΛG acts on the left side. Every ΛG-module M can be viewed as a ΛH-module. We denote this restricted module by $M \downarrow_H^G$. Observe that ΛG is a free ΛH-module. Therefore, these two functors are exact, and perverse projective modules.

The following property can be deduced from Lemmas 2.3 and 2.4 in [4]. Here we give an elementary proof.

Proposition 2.1. *Let V and M be a ΛH-module and a ΛG-module respectively. Then:*

(1) *V is a summand of $V \uparrow_H^G \downarrow_H^G$;*
(2) *if $|G : H|$ is invertible in k, then M is a summand of $M \downarrow_H^G \uparrow_H^G$.*

Proof. Let $\{g_1 = 1_G, g_2, \ldots, g_n\}$ be a set of right coset representatives. As vector spaces, we have

$$V \uparrow_H^G = \Lambda G \otimes_{\Lambda H} V = \bigoplus_{i=1}^{n} g_i \otimes_{\Lambda H} V.$$

Let $\tilde{V} = \bigoplus_{i=2}^{n} g_i \otimes_{\Lambda H} V$. Then

$$V \uparrow_H^G \downarrow_H^G = 1_G \otimes_{\Lambda H} V \oplus \tilde{V} \cong V \oplus \tilde{V}.$$

This is actually a decomposition of ΛH-modules, and (1) is proved.

If $n = |G : H|$ is invertible, we can define the following maps:

$$\theta : \Lambda G \otimes_{\Lambda H} M \to M, \quad \lambda g \otimes v \mapsto \lambda g(v);$$

$$\eta : M \to \Lambda G \otimes_{\Lambda H} M, \quad v \mapsto \frac{1}{n} \sum_{i=1}^{n} g_i^{-1} \otimes g_i(v).$$

As for group algebras, we can check that these two maps are ΛG-module homomorphisms, and $\theta\eta$ is the identity map from M to M. For example, for all $\lambda \in \Lambda$, $g \in G$ and $v \in M$, we have:

$$\begin{aligned}\eta(\lambda g v) &= \frac{1}{n} \sum_{i=1}^{n} g_i^{-1} \otimes g_i(\lambda g(v)) = \frac{1}{n} \sum_{i=1}^{n} g_i^{-1} \otimes ({}^{g_i}\lambda)(g_i g)(v) \\ &= \frac{1}{n} \sum_{i=1}^{n} g_i^{-1}({}^{g_i}\lambda) \otimes (g_i g)(v) = \frac{1}{n} \sum_{i=1}^{n} \lambda g g^{-1} g_i^{-1} \otimes (g_i g)(v) \\ &= (\lambda g)(\frac{1}{n} \sum_{i=1}^{n} (g_i g)^{-1} \otimes (g_i g)(v)) = \lambda g(\eta(v)).\end{aligned}$$

This proves (2). $\square$

The above proposition immediately implies:

Corollary 2.2. *Let $H \leqslant G$ be a subgroup of G and M be a ΛG-module.*

(1) *If ΛG is of finite representation type, so is ΛH, and in particular, Λ is of finite representation type.*
(2) *If the global dimension* $\operatorname{gldim} \Lambda G < \infty$, *then* $\operatorname{gldim} \Lambda H < \infty$, *and in particular,* $\operatorname{gldim} \Lambda < \infty$.
(3) *If Q is an injective ΛG-module, then $Q \downarrow_H^G$ is an injective ΛH-module.*

Suppose that $|G : H|$ is invertible. Then:

(4) *ΛG is of finite representation type if and only if so is ΛH.*
(5) *M is a projective ΛG-module if and only if $M \downarrow_H^G$ is a projective ΛH-module. In particular,* $\operatorname{gldim} \Lambda G = \operatorname{gldim} \Lambda H$.
(6) *Q is an injective ΛG-module if and only if $Q \downarrow_H^G$ is an injective ΛH-module.*

Proof. The proof of these facts are straightforward.

(1): By the first statement of the above proposition, every indecomposable ΛH-module is a summand of the restricted module of an indecomposable ΛG-module. Therefore, the finite representation type of ΛG implies the finite representation type of ΛH. In particular, taking $H = 1$, we conclude that if ΛG is of finite representation type, so is Λ. This proves (1).

(2): Suppose that $\operatorname{gldim} \Lambda G = n < \infty$. For every ΛH-module V, $V \uparrow_H^G$ has a projective resolution whose length is bounded by n. Applying the restriction functor we get a projective resolution of $V \uparrow_H^G \downarrow_H^G$. Thus $\operatorname{pd}_{\Lambda H} V \uparrow_H^G \downarrow_H^G \leqslant n$. Since V is a summand of $V \uparrow_H^G \downarrow_H^G$, we deduce that $\operatorname{pd}_{\Lambda H} V \leqslant n$ as well. Therefore, $\operatorname{gldim} \Lambda H \leqslant n$. In particular, $\operatorname{gldim} \Lambda$ is finite.

(3): The ΛH-module $Q \downarrow_H^G$ is injective if and only if the functor $\operatorname{Hom}_{\Lambda H}(-, Q \downarrow_H^G)$ is exact. By the Nakayama relations, this is true if and only if $\operatorname{Hom}_{\Lambda G}(- \uparrow_H^G, Q)$, the composite of the induction functor and $\operatorname{Hom}_{\Lambda G}(-, Q)$, is exact, which is obvious since both functors are exact.

Now suppose that $n = |G : H|$ is invertible in k.

(4): By the second statement of the above proposition, every indecomposable ΛG-module is a summand of the induced module of an indecomposable ΛH-module. Therefore, the finite representation type of ΛH implies the finite representation type of ΛG.

(5): Since induction and restriction functors perverse projective modules, and M is a summand of $M \downarrow_H^G \uparrow_H^G$, we know that M is a projective ΛG-module if $M \downarrow_H^G$ a projective ΛH-module. The other direction is trivially true.

We already showed that $\operatorname{gldim} \Lambda H \leqslant \operatorname{gldim} \Lambda G$. Now let $P^\bullet$ be a projective resolution of $M \downarrow_H^G$. Applying the induction functor we get a projective resolution $P^\bullet \uparrow_H^G$ of $M \downarrow_H^G \uparrow_H^G$, which contains M as a summand. This immediately implies $\operatorname{gldim} \Lambda G \leqslant \operatorname{gldim} \Lambda H$. Therefore, $\operatorname{gldim} \Lambda G = \operatorname{gldim} \Lambda H$.

(6): The ΛG-module Q is injective if and only if $\operatorname{Hom}_{\Lambda G}(-, Q)$ is exact. Note that this functor is a summand of the functor $\operatorname{Hom}_{\Lambda G}(- \downarrow_H^G \uparrow_H^G, Q)$ since $|G : H|$ is invertible in k. Therefore, it suffices to show the exactness of $\operatorname{Hom}_{\Lambda G}(- \downarrow_H^G \uparrow_H^G, Q)$. By the Nakayama relations, it is isomorphic to $\operatorname{Hom}_{\Lambda H}(- \downarrow_H^G, Q \downarrow_H^G)$, the composite functor of $\downarrow_H^G$ and $\operatorname{Hom}_{\Lambda H}(-, Q \downarrow_H^G)$. The conclusion follows from the observation that the restriction functor is exact and $Q \downarrow_H^G$ is injective. □

In the case that kG is semisimple, we get the following well known result.

Corollary 2.3. *If $|G|$ is invertible in k, then ΛG is of finite representation type if and only if so is Λ; a ΛG-module M is projective if and only if it is a projective Λ-module; and* $\operatorname{gldim} \Lambda G = \operatorname{gldim} \Lambda$.

Proof. Since $|G|$ is invertible, $S = 1$, and hence $\Lambda S \cong \Lambda$. The conclusion follows immediately from the previous corollary. □

3. Actions of p-groups

From the previous section we know that the action of a Sylow p-subgroup $S \leqslant G$ plays a crucial role in determining the representation type and the global dimension of ΛG. Explicitly, ΛG and ΛS have the same representation type and the same global dimension. Throughout this section let S be a p-group. We suppose that there is set $E = \{e_i\}_{i=1}^n$ of primitive orthogonal idempotents in Λ which is *complete* (i.e., $\sum_{i=1}^n e_i = 1$) and is *closed* under the action of S. Although this is not always true as explained in the following example, in many cases the action of S on Λ is induced by an action of S on some discrete structures such as quivers, posets, etc, and this condition is satisfied.

Example 3.1. *Let Λ be the path algebra of the following quiver with relations $\alpha\beta = \beta\alpha = 0$.*

$$\begin{tikzcd} x \arrow[r, bend left, "\alpha"] & y \arrow[l, bend left, "\beta"] \end{tikzcd}.$$

Let $S = \langle g \rangle$ be a cyclic group of order 2. Let k be an algebraically closed field of characteristic 2. We define an action of g on Λ as follows: $g(1_x) = 1_x + \alpha$, $g(\alpha) = \alpha$, $g(1_y) = 1_y + \beta$ and $g(\beta) = \beta$. This action gives rise to an algebra automorphism of Λ. However, we can check that there does not exist a complete set of primitive orthogonal idempotents closed under the action of g.

Lemma 3.2. *The set E is also a complete set of primitive orthogonal idempotents of ΛS, where we use e_i rather than $e_i 1_S$ to simplify the notation.*

Proof. If $\sum_{g\in S}\lambda_g g\in \Lambda S$ is an idempotent, then

$$\sum_{g\in S}\lambda_g g=(\sum_{g\in S}\lambda_g g)^{|S|}=\sum_{g\in S}(\lambda_g g)^{|S|}$$

has the form of $\mu 1_S$, $\mu\in\Lambda$. From $\mu 1_S=(\mu 1_S)^2$ we know that μ is an idempotent in Λ. Therefore, the identity element of ΛS can be expressed as a sum of at most n primitive orthogonal idempotents. Consequently, we can choose E as a complete set of primitive orthogonal idempotents of ΛS. $\square$

From this lemma we know that every indecomposable ΛS-module can be described as follows (up to isomorphism):

$$\Lambda S e_i=\bigoplus_{g\in S}\Lambda g(e_i)\otimes_k g,\quad 1\leqslant i\leqslant n.$$

Since E is closed under the action of S, we take $\{e_1,e_2,\ldots,e_s\}$ to be a set of representatives of S-orbits, and set $\epsilon=\sum_{i=1}^s e_i$. This is an idempotent of Λ, and hence an idempotent of ΛS.

The next lemma is described in [6] for the situation that S acts freely on E, see Lemma 3.1 in that paper.

Lemma 3.3. *Let S,Λ,E and ϵ be as defined before. Then ΛS is Morita equivalent to $\epsilon\Lambda\epsilon$.*

Proof. If e_i and e_j are in the same S-orbit, we can find some $1_S\neq g\in S$ such that $g(e_i)=ge_ig^{-1}=e_j$. It is straightforward to check that for every e_s, $1\leqslant s\leqslant n$,

$$\begin{aligned} e_j(\Lambda S)e_s &= ge_ig^{-1}(\Lambda S)e_s=ge_ig^{-1}(\Lambda S)gg^{-1}e_s\\ &= ge_i(\Lambda S)g^{-1}(e_s)g^{-1}\cong e_i(\Lambda S)g^{-1}(e_s). \end{aligned}$$

In particular, let $s=j$, we have

$$e_j(\Lambda S)e_j\cong e_i(\Lambda S)g^{-1}(e_j)=e_i(\Lambda S)e_i.$$

Therefore, if e_i and e_j are in the same S-orbit, then $\Lambda Se_i\cong\Lambda Se_j$ as ΛS-modules.

Actually, if we consider the associated k-linear category $\mathcal{A}$ of ΛS defined by Gabriel's construction ([2]), then two idempotents in the same S-orbit are isomorphic when viewed as objects in $\mathcal{A}$. Therefore, ΛS is Morita equivalent to the skeletal category of $\mathcal{A}$, and this skeletal category is Morita equivalent to $\epsilon\Lambda S\epsilon$ by Gabriel's construction again. $\square$

Although we assumed in this lemma that S is a p-group, the conclusion holds for arbitrary groups.

The following proposition motivates us to study the situation that S acts freely on E. That is, for every $1\leqslant i\leqslant n$ and $g\in S$, $g(e_i)\neq e_i$.

Proposition 3.4. *Let S,Λ and E be as above.*

1. *If* $\operatorname{gldim}\Lambda S<\infty$, *then the action of S on E is free.*
2. *Suppose that $p\neq 2,3$ and Λ is not a local algebra. If ΛS is of finite representation type, then S acts freely on E.*

Proof. We use contradiction to prove the first part. Without loss of generality we assume that there is some $1_S\neq g\in S$ such that $g(e_1)=e_1$. Consider the subgroup $H=\langle g\rangle\leqslant S$. By the assumption, H is a nontrivial p-group. By Corollary 2.2, we only need to show $\operatorname{gldim}\Lambda H=\infty$.

It is obvious that $\Lambda H e_1$ is an indecomposable projective ΛH-module. Moreover, since $H = \langle g \rangle$ and g fixes e_1, we actually have $\Lambda H e_1 = \Lambda e_1 H = \Lambda e_1 \otimes_k kH$ as ΛH-modules, where $h \in H$ acts diagonally on $\Lambda e_1 \otimes_k kH$. Consider the ΛH-module $\Lambda e_1 \cong \Lambda e_1 \otimes_k k$, where $h \in H$ acts on k trivially and sends $\lambda e_1 \in \Lambda e_1$ to $h(\lambda) e_1$. It has a projective cover $\Lambda e_1 \otimes_k kH$ and the first syzygy is isomorphic to $\Lambda e_1 \otimes_k \mathfrak{r}$, where $\mathfrak{r}$ is the radical of kH. Clearly, the first syzygy is not projective and its projective cover is isomorphic to $\Lambda e_1 \otimes_k kH$. Observe that $\Lambda e_1 \otimes_k k$ appears again as the second syzygy. Consequently, $\Lambda e_1 \otimes_k k$ has infinite projective dimension, so $\operatorname{gldim} \Lambda H = \infty$. This contradiction establishes the conclusion.

Now we prove the second part by contradiction. Since it is obviously true if S is a trivial group (for example, if $p = 0$), without loss of generality we assume $S \neq 1$, so $p \geqslant 5$. Suppose that there are a primitive idempotent e_i and some $1_S \neq g \in S$ such that $g(e_i) = e_i$. We want to show that ΛS is of infinite representation type. Let H be the cyclic group generated by g and consider ΛH. By Corollary 2.2, it suffices to prove that ΛH is of infinite representation type.

Since Λ is a connected algebra and is not local, we can find another primitive idempotent e_j connected to e_i, i.e., either $e_i \Lambda e_j \neq 0$ or $e_j \Lambda e_i \neq 0$. Consider the algebra $\Gamma = (e_i + e_j) \Lambda H (e_i + e_j)$. By Proposition 2.5 in page 35 of [1], Γ-mod is equivalent to a subcategory of ΛH-mod. Therefore, we only need to show the infinite representation type of Γ.

Note that $\Lambda H e_i$ and $\Lambda H e_j$ are not isomorphic as ΛH-modules. Indeed, if they are isomorphic as ΛH-modules, then viewed as Λ-modules they must be isomorphic as well. But ${}_\Lambda \Lambda H e_i \cong (\Lambda e_i)^{|H|}$ since H fixes e_i. On the other hand, ${}_\Lambda \Lambda H e_j \cong \bigoplus_{h \in H} \Lambda h(e_j)$. For every $h \in H$, $h(e_j) \neq e_i$ since otherwise we should have $h^{-1}(e_i) = e_j$. This is impossible since H fixes e_i. Therefore, ${}_\Lambda \Lambda H e_i$ is not isomorphic to ${}_\Lambda \Lambda H e_j$ and hence $\Lambda H e_i \ncong \Lambda H e_j$. Correspondingly, Γ is a basic algebra with two indecomposable projective summands.

Now let us study the structure of Γ. Firstly, $\dim_k e_i \Lambda H e_i \geqslant |H| \geqslant p \geqslant 5$ since it contains linearly independent vectors $e_i h = h e_i$ for every $h \in H$. Furthermore,

$$e_i \Lambda H e_j = \bigoplus_{h \in H} e_i \Lambda h(e_j) \otimes_k h, \quad e_j \Lambda H e_i = \bigoplus_{h \in H} e_j \Lambda e_i \otimes_k h.$$

Since h fixes e_i and sends e_j to $h(e_j)$, and e_j is connected to e_i, we deduce that either $\dim_k e_i \Lambda h(e_j) = \dim_k e_i \Lambda e_j \geqslant 1$ or $\dim_k e_j \Lambda e_i \geqslant 1$. Therefore, either $\dim_k e_i \Lambda H e_j \geqslant |H| \geqslant p \geqslant 5$ or $\dim_k e_j \Lambda H e_i \geqslant |H| \geqslant p \geqslant 5$.

In [2, 5] the representation types of algebras with two nonisomorphic indecomposable projective modules have been classified. By the list described in these papers, we conclude that Γ is of finite representation type only if it is isomorphic to the following algebra R with relations $(\alpha\beta)^t = (\beta\alpha)^t = 0$ for some $t \geqslant p \geqslant 5$:

$$\begin{tikzcd} e_i \arrow[r, bend left, "\alpha"] & e_j \arrow[l, bend left, "\beta"] \end{tikzcd}.$$

We claim that this is impossible. Indeed, if $\Gamma \cong R$, then every non-invertible element in $e_i \Gamma e_i$ is contained in $e_i \Gamma e_j \Gamma e_i = e_i \Lambda H e_j \Lambda H e_i$ since R has this property. In particular, $e_i (g - 1_H) \in e_i \Gamma e_i$, which is not invertible, is contained in $e_i \Lambda H e_j \Lambda H e_i$.

We have:

$$\begin{aligned} e_i \Lambda H e_j \Lambda H e_i &= e_i\Big(\bigoplus_{h \in H} \Lambda h(e_j) h\Big)(\Lambda H e_i) \subseteq \bigoplus_{h \in H} e_i \Lambda h(e_j) h (\Lambda H e_i) \\ &\subseteq \bigoplus_{h \in H} e_i \Lambda h(e_j)(\Lambda H e_i) = \bigoplus_{h \in H} (e_i \Lambda h(e_j) \Lambda e_i) H \end{aligned}$$

since H fixes e_i. Note that $h(e_j) \neq e_i$ for every $h \in H$, so $e_i \Lambda h(e_j) \Lambda e_i \subseteq \operatorname{rad}(e_i \Lambda e_i)$. Thus $e_i(g - 1_H)$ is contained in $e_i \Lambda H e_j \Lambda H e_i \subseteq \operatorname{rad}(e_i \Lambda e_i) H$, and we can write

$$e_i(g - 1_H) = \sum_{h \in H} \lambda_h h = \lambda_{1_H} 1_H + \sum_{1_H \neq h \in H} \lambda_h h$$

where $\lambda_h \in \operatorname{rad}(e_i \Lambda e_i)$. Therefore,

$$(-e_i + \lambda_{1_H}) 1_H = \sum_{1_H \neq h \in H} \lambda_h h - e_i g.$$

This happens if and only if $e_i - \lambda_{1_H} = 0$. But this is impossible since e_i is the identity element of $e_i \Lambda e_i$ and $\lambda_{1_H} \in \operatorname{rad}(e_i \Lambda e_i)$. Therefore, as we claimed, Γ cannot be isomorphic to the path algebra R of the above quiver, so it must be of infinite representation type. □

Although we have assumed the condition that Λ has a complete set of primitive orthogonal idempotents which is closed under the action of S, it can be replaced by a slightly weaker condition that Λ can be decomposed into a set of indecomposable summands which is closed under the action of S. Indeed, in Example 2.1 $\Lambda = \Lambda 1_x \oplus \Lambda 1_y$, and

$$g(\Lambda 1_x) = \Lambda g(1_x) = \Lambda(1_x + \alpha) = \Lambda 1_x, \quad g(\Lambda 1_y) = \Lambda g(1_y) = \Lambda(1_y + \alpha) = \Lambda 1_y.$$

With some small modifications, we can show that the above proposition is still true under this weaker condition.

In next section we will use the second statement of this proposition to classify the representation types of skew groups algebras ΛG for $p \neq 2, 3$ when Λ is the incidence algebra of a finite connected poset $\mathcal{P}$ and elements of G act as automorphisms on $\mathcal{P}$.

We reminder the reader that the condition that $p \neq 2, 3$ is required, as shown by the following example.

Example 3.5. *Let Λ be the path algebra of the quiver $x \to y$. Let $S = \langle g \rangle$ be a cyclic group of order $p = 2$. Suppose that the action of S on Λ is trivial. That is, $g(x) = x$ and $g(y) = y$. Then ΛS is isomorphic to the path algebra of the following quiver with relations $\alpha\delta = \theta\alpha$, $\theta^2 = \delta^2 = 0$. By Bongartz's list in [2], this algebra is of finite representation type. However,* $\operatorname{gldim} \Lambda S = \infty$ *although* $\operatorname{gldim} \Lambda = 1$.

$$\begin{tikzcd} \mathcal{E}: & x \arrow[loop left, "\delta"] \arrow[r, "\alpha"] & y \arrow[loop right, "\theta"] \end{tikzcd}.$$

Now let Λ be the path algebra of the following quiver such that the composite of every two arrows is 0. Let $S = \langle g \rangle$ be a cyclic group of order $p = 5$. The action of g on Λ is determined by $g(i) = (i + 1) \mod 5$. Then we find all idempotents are isomorphic in ΛS. Consequently, ΛS is isomorphic to the algebra of 5×5 matrices over $k[X]/(X^2)$. It is Morita equivalent to $k[X]/(X^2)$, which is of finite representation type. It is also clear that both Λ and ΛS have infinite global dimension.

Moreover, Λ^S is spanned by 1 and $\alpha+\beta+\gamma+\delta+\theta$, and $\Lambda^S \cong k[X]/(X^2)$.

$$\begin{tikzcd}
 & 1 \arrow[dr, "\alpha"] & \\
5 \arrow[ur, "\theta"] & & 2 \arrow[d, "\beta"] \\
4 \arrow[u, "\delta"] & & 3 \arrow[ll, "\gamma"]
\end{tikzcd}$$

In the rest of this section we consider in details the situation that S acts freely on E. Then $|S|$ divides n, so E has $s = n/|S|$ S-orbits. We will show that this situation generalize many results in the case that a finite group of invertible order acts on Λ (note in this special case $S = 1$ always acts freely on E). In particular, the radical of ΛS is $\mathfrak{r}S$, and a ΛS-module is projective if and only if viewed as a Λ-module it is projective.

Let M be a ΛS-module, the elements $v \in M$ satisfying $g(v) = v$ for every $g \in S$ form a ΛS-submodule, which is denoted by M^S. Let F^S be the endo-functor from ΛS-mod, the category of finitely generated ΛS-modules, to itself, sending M to M^S. This is indeed a functor since for every ΛS-homomorphism $f : M \to N$, M^S is mapped into N^S by f.

Proposition 3.6. *Let S, Λ, E and F^S be as before and suppose that S acts freely on E. We have:*

1. *(1) The i-th radical of ΛS is $\mathfrak{r}^i S$, where $\mathfrak{r}$ is the radical of Λ, $i \geqslant 1$.*
2. *(2) The functor F^S is exact.*
3. *(3) The regular representation ${}_{\Lambda S}\Lambda S \cong \Lambda^{|S|}$, where Λ is the trivial ΛS-module.*
4. *(4) The basic algebra of ΛS is isomorphic to $\operatorname{End}_{\Lambda S}(\Lambda) \cong \Lambda^S$, the space constituted of all elements in Λ fixed by S.*
5. *(5) A ΛS-module M is projective (resp., injective) if and only if the Λ-module ${}_{\Lambda}M$ is projective (resp., injective). In particular, ΛS is self-injective if and only if so is Λ.*

Proof. Since the action of every $g \in S$ on Λ is an algebra automorphism, it sends $\mathfrak{r}$ onto $\mathfrak{r}$. Therefore, $\mathfrak{r}S = \mathfrak{r} \otimes_k kS$ is a ΛS-module. Actually, it is a two-sided ideal of ΛS. Moreover, this ideal is nilpotent since so is $\mathfrak{r}$. Therefore, $\mathfrak{r}S$ is contained in $\mathfrak{R}$, the radical of ΛS.

On the other hand, $\Lambda S e_i \cong \Lambda S g(e_i)$ for $g \in S$. Correspondingly, the dimension of every simple ΛS-module is at least $|S|$. Furthermore, ΛS and Λ have the same idempotents by identifying $e_i 1_S$ with e_i. Therefore, $\Lambda S/\mathfrak{R}$ has n simple summands, each of which has dimension at least $|S|$. Thus $\dim_k \Lambda S/\mathfrak{R} \geqslant n|S|$. We also have $\dim_k \Lambda S/\mathfrak{r}S = \dim_k(\Lambda/\mathfrak{r}) \times |S|$. But $\Lambda/\mathfrak{r}$ is a semisimple Λ-module with n simple summands, each of which has dimension 1. Therefore, $\dim_k(\Lambda S)/\mathfrak{r}S = n|S|$. This forces $\mathfrak{R} = \mathfrak{r}S$, $\mathfrak{R}^2 = \mathfrak{r}S\mathfrak{r}S = \mathfrak{r}^2 S$, and so on. The first part is proved.

Take an arbitrary short exact sequence of ΛS-modules

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\theta} N \longrightarrow 0 .$$

Applying the functor F^S we get a sequence

$$0 \longrightarrow L^S \xrightarrow{\varphi'} M^S \xrightarrow{\theta'} N^S \longrightarrow 0 ,$$

where φ' and θ' are $F^S(\varphi)$ and $F^S(\theta)$, the restrictions of φ and θ to L^S and M^S respectively. It is clear that φ' is injective. Therefore, we only need to show that θ' is surjective. Take $w \in N$. We can find some $v \in M$ such that $\theta(v) = w$.

Recall ϵ is the sum of $s = n/|S|$ primitive idempotents in E, each of which comes from a distinct S-orbit. Since S acts on E freely, $1 = \sum_{g \in S} g(\epsilon)$. Now define $\tilde{v} = \sum_{g \in S} g(\epsilon v) = \sum_{g \in S} g(\epsilon) g(v)$. We check that $\tilde{v} \in M^S$, and

$$\begin{aligned} \theta'(\tilde{v}) &= \theta(\tilde{v}) = \theta(\sum_{g \in S} g(\epsilon v)) = \sum_{g \in S} \theta(g(\epsilon v)) = \sum_{g \in S} g(\epsilon)\theta(g(v)) \\ &= \sum_{g \in S} g(\epsilon) g(\theta(v)) = \sum_{g \in S} g(\epsilon) g(w) = \sum_{g \in S} g(\epsilon) w = w \end{aligned}$$

where $g(w) = w$ since $w \in N^S$. This shows that θ' is surjective, and establishes the second part.

As ΛS-modules, $\Lambda S = \bigoplus_{g \in S} \Lambda S g(\epsilon)$. Note that ϵ and $g(\epsilon)$ are isomorphic idempotents for every $g \in S$. Therefore, $\Lambda S \cong (\Lambda S \epsilon)^{|S|}$. We deduce that ΛS has precisely $s = n/|S|$ non-isomorphic indecomposable projective modules $\Lambda S e_i$, $1 \leqslant i \leqslant s$ since $\epsilon = e_1 + \ldots + e_s$.

Since $\sum_{g \in S} g(\epsilon) = 1$, by Proposition 4.1 in page 87 of [1], the trivial ΛS-module ${}_{\Lambda S}\Lambda$ is projective. Moreover, distinct indecomposable summands P and Q of ${}_{\Lambda S}\Lambda$ are non-isomorphic since otherwise viewed as Λ-modules P and Q must be isomorphic as well. But this is impossible since Λ is a basic algebra. Consequently, ${}_{\Lambda S}\Lambda$ can have at most s indecomposable summands. On the other hand, it is clear that $\Lambda \sum_{g \in S} g(e_i)$ is a submodule of ${}_{\Lambda S}\Lambda$, and ${}_{\Lambda S}\Lambda = \bigoplus_{i=1}^{s} (\Lambda \sum_{g \in S} g(e_i))$. Therefore, the projective module ${}_{\Lambda S}\Lambda$ has at least s indecomposable summands. This forces ${}_{\Lambda S}\Lambda$ to have precisely s indecomposable summands, and all of them are pairwise non-isomorphic. Consequently, ${}_{\Lambda S}\Lambda \cong \Lambda S \epsilon$, and

$${}_{\Lambda S}\Lambda S \cong (\Lambda S \epsilon)^{|S|} \cong ({}_{\Lambda S}\Lambda)^{|S|}.$$

This proves the third statement, and forth one follows immediately.

Now we turn to the last statement. The ΛS-module M is projective if and only if the functor $\mathrm{Hom}_{\Lambda S}(M, -)$ is exact. But $\mathrm{Hom}_{\Lambda S}(M, -) = \mathrm{Hom}_{\Lambda}(M, -)^S$ is a composite of $\mathrm{Hom}_{\Lambda}(M, -)$ and the exact functor F^S. Therefore, $\mathrm{Hom}_{\Lambda S}(M, -)$ is exact if and only if $\mathrm{Hom}_{\Lambda}(M, -)$ is exact, which is equivalent to saying that the restricted module ${}_{\Lambda}M$ is projective. Using the same technique we can show that M is injective if and only if so is ${}_{\Lambda}M$.

Suppose that Λ is self-injective and let P be a projective ΛS-module. Then ${}_{\Lambda}P$ is a projective Λ-module, so is injective. Thus P is also injective. It immediately follows that ΛS is self-injective.

Conversely, suppose that ΛS is self-injective. Take an arbitrary projective Λ-module Q and consider $Q \uparrow_1^S = \Lambda S \otimes_{\Lambda} Q$. It is projective, so is injective as well. Therefore, $Q \uparrow_1^S \downarrow_1^S$ is also injective. As a summand of $Q \uparrow_1^S \downarrow_1^S$, Q is also injective. Therefore, Λ is self-injective. □

To prove the first main result, we want to show that the finite representation type of Λ implies the finite representation type of Λ^S when S acts freely on E. Firstly, we prove the following lemmas.

Lemma 3.7. *Suppose that S acts freely on E. Then $\Lambda^S = \{\sum_{g \in S} g(\mu) \mid \mu \in \Lambda\}$.*

Proof. Take $\lambda \in \Lambda^S$. Recall we have chosen $\{e_1, e_2, \dots, e_s\}$ to be a set of representatives of S-orbits and define $\epsilon = \sum_{i=1}^{s} e_i$. Then $\lambda = \sum_{g\in S} \lambda g(\epsilon) = \sum_{g\in S} g(\lambda\epsilon)$ since $g(\lambda) = \lambda$. Let $\mu = \lambda\epsilon$ we get the conclusion. □

Clearly, Λ^S is a subalgebra of Λ, and S acts on it trivially. For every Λ^S-module M, we can define a Λ-module $\Lambda \otimes_{\Lambda^S} M$. The next lemma tells us that M is a summand of $(\Lambda \otimes_{\Lambda^S} M) \downarrow^{\Lambda}_{\Lambda^S}$.

We first define two linear maps:

$$\theta : M \to (\Lambda \otimes_{\Lambda^S} M) \downarrow^{\Lambda}_{\Lambda^S}, \quad v \mapsto \sum_{g\in S} g(\epsilon) \otimes v, \quad v \in M;$$

$$\varphi : (\Lambda \otimes_{\Lambda^S} M) \downarrow^{\Lambda}_{\Lambda^S} \to M, \quad \lambda \otimes v \mapsto (\sum_{g\in S} g(\lambda\epsilon))v, \quad v \in M, \lambda \in \Lambda.$$

Since $\sum_{g\in S} g(\lambda\epsilon) \in \Lambda^S$, its action on v is well defined. Moreover, we observe that when $\lambda \in \Lambda^S$,

$$\varphi(\lambda \otimes v) = \big(\sum_{g\in S} g(\lambda\epsilon)\big)v = \big(\sum_{g\in S} \lambda g(\epsilon)\big)v = \lambda v.$$

Lemma 3.8. *Notation as above. Both φ and θ are Λ^S-module homomorphisms. Moreover, $\varphi\theta$ is the identity map on M. In particular, if Λ is of finite representation type, so is Λ^S.*

Proof. For every $u \in \Lambda^S$, $\lambda \in \Lambda$ and $v \in M$, we have

$$\theta(\mu\lambda \otimes v) = \big(\sum_{g\in S} g(\mu\lambda\epsilon)\big)v = \big(\sum_{g\in S} \mu g(\lambda\epsilon)\big)v = \mu\theta(\lambda \otimes v).$$

Thus θ is indeed a Λ^S-module homomorphism.

Now we show that φ is a Λ^S-module homomorphism. Take $\lambda \in \Lambda^S$. By the previous lemma, λ can be expressed as $\sum_{g\in S} g(\mu)$ for some $\mu \in \Lambda$. Since φ is clearly a linear map, we only need to deal with μ supported on two primitive idempotents. That is, there are $1 \leqslant i, j \leqslant n$ such that $\mu = e_i \mu e_j$. Since each S-orbit of E has a representative as a component of ϵ, there is a unique $x \in S$ such that $\epsilon x(e_i) = x(e_i)$ and a unique $y \in S$ such that $y(e_j)\epsilon = y(e_j)$. Consequently, $\epsilon\lambda = x(\mu)$ and $\lambda\epsilon = y(\mu)$. Therefore, we have:

$$\epsilon\lambda = x(\mu) = xy^{-1}(y(\mu)) = xy^{-1}(\lambda\epsilon),$$

and

$$\begin{aligned}\varphi(\lambda v) &= \sum_{g\in S} g(\epsilon) \otimes \lambda v = \sum_{g\in S} g(\epsilon)\lambda \otimes v = \sum_{g\in S} g(\epsilon\lambda) \otimes v \\ &= \sum_{g\in S} (gxy^{-1})(\lambda\epsilon) \otimes v = \lambda \sum_{g\in S} (gxy^{-1})(\epsilon) \otimes v = \lambda\varphi(v).\end{aligned}$$

Thus φ is also a Λ^S-module homomorphism.

For every $v \in M$, $\theta\varphi(v) = \theta(\sum_{g\in S} g(\epsilon) \otimes v)$. Note that $\sum_{g\in S} g(\epsilon) \in \Lambda^S$. By the remark before this lemma,

$$\theta(\varphi(v)) = \theta(\sum_{g\in S} g(\epsilon) \otimes v) = (\sum_{g\in S} g(\epsilon))v = v.$$

Indeed, $\theta\varphi$ is the identity map on M. Therefore, M is a summand of $(\Lambda \otimes_{\Lambda^S} M) \downarrow^{\Lambda}_{\Lambda^S}$. Consequently, every indecomposable Λ^S-module is a summand of the restricted

module of an indecomposable Λ-module, so the finite representation type of Λ implies the finite representation type of Λ^S. $\square$

Now we are ready to prove the first main result. Recall an algebra A is called an *Auslander algebra* if $\operatorname{gldim} A \leqslant 2$, and if

$$0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow 0$$

is a minimal injective resolution, then I_0 and I_1 are projective as well.

Theorem 3.9. *Let Λ be a basic finite-dimensional k-algebra and G be a finite group acting on Λ. Let $S \leqslant G$ be a Sylow p-subgroup. Suppose that there is a set of primitive orthogonal idempotents $E = \{e_i\}_{i=1}^n$ such that E is closed under the action of S and $\sum_{i=1}^n e_i = 1$. Then:*

(1) $\operatorname{gldim} \Lambda G < \infty$ if and only if $\operatorname{gldim} \Lambda < \infty$ and S acts freely on E. Moreover, if $\operatorname{gldim} \Lambda G < \infty$, then $\operatorname{gldim} \Lambda G = \operatorname{gldim} \Lambda$.

(2) ΛG is an Auslander algebra if and only if so is Λ and S acts freely on E.

(3) Suppose that $p \neq 2, 3$ and Λ is not a local algebra. Then ΛG is of finite representation type if and only if S acts freely on E, and Λ is of finite representation type.

Proof. By Corollary 2.2, we only need to deal with the case that G is a p-group, i.e., $G = S$.

(1): The only if part of the first statement follows from the second statement of Corollary 2.2 and Lemma 3.4. Now we prove the if part.

Suppose that $G = S$ acts freely on E and $\operatorname{gldim} \Lambda = l < \infty$. For an arbitrary ΛS-module M, take a projective resolution

$$\ldots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow M.$$

Applying the restriction functor we get a projective resolution

$$\ldots \longrightarrow {}_\Lambda P^1 \longrightarrow {}_\Lambda P^0 \longrightarrow {}_\Lambda M.$$

Therefore, for $i \geqslant 0$, ${}_\Lambda\Omega^i(M)$ is a direct sum of $\Omega^i({}_\Lambda M)$ and a projective Λ-module. Since $\operatorname{gldim} \Lambda = l < \infty$, $\Omega^l({}_\Lambda M) = 0$, so ${}_\Lambda\Omega^l(M)$ is a projective Λ-module. By (5) of Proposition 3.6, $\Omega^l(M)$ must be projective. Thus $\operatorname{gldim} \Lambda S \leqslant \operatorname{gldim} \Lambda < \infty$.

It is always true that $\operatorname{gldim} \Lambda S \geqslant \operatorname{gldim} \Lambda$. In the above proof we have actually shown that the other inequality is also true if $\operatorname{gldim} \Lambda < \infty$ and S acts freely on E. This completes the proof of (1)

(2): If Λ is an Auslander algebra, then $\operatorname{gldim} \Lambda \leqslant 2$. Since G acts freely on E, we know $\operatorname{gldim} \Lambda G = \operatorname{gldim} \Lambda \leqslant 2$ by (1). Moreover, the minimal injective resolution of ΛG-modules

$$0 \longrightarrow \Lambda G \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow 0$$

gives rises to

$$0 \longrightarrow {}_\Lambda\Lambda G \longrightarrow {}_\Lambda I_0 \longrightarrow {}_\Lambda I_1 \longrightarrow {}_\Lambda I_2 \longrightarrow 0$$

which contains a minimal injective resolution of Λ as a summand since ${}_\Lambda\Lambda G \cong \Lambda^{|G|}$. Since ${}_\Lambda I_0$ and ${}_\Lambda I_1$ are both injective and projective, By (5) of Proposition 3.6, I_0 and I_1 are both projective and injective. Therefore, ΛG is an Auslander algebra.

Conversely, if ΛG is an Auslander algebra, then $\operatorname{gldim} \Lambda G \leqslant 2$. Therefore, $\operatorname{gldim} \Lambda \leqslant 2$ and G must act freely on E by (1). As explained in the previous

paragraph, a minimal injective resolution of ΛG gives rise to a minimal injective resolution of Λ satisfying the required condition. Thus Λ is an Auslander algebra.

(3): By the second part of Proposition 3.4, if ΛS is of finite representation type, then the action of S on E must be free; by Corollary 2.2, Λ must be of finite representation type. Conversely, if S acts freely on E, then by (4) of Proposition 3.6, ΛS is Morita equivalent to Λ^S. By the previous lemma, Λ^S is of finite representation type since so is Λ. $\square$

4. Representation types of transporter categories

Throughout this section let $\mathcal{P}$ be a connected finite poset with a partial order $\leqslant$ and G be a finite group of automorphisms on $\mathcal{P}$. We consider the *transporter category* $\mathcal{T} = G \propto \mathcal{P}$. The category $\mathcal{T}$ has the same objects as $\mathcal{P}$ (viewed as a finite category). All morphisms in $\mathcal{T}$ are of the form αg, where $g \in G$ and $\alpha : x \to y$ is a morphism in $\mathcal{P}$ (which means $x \leqslant y$ in $\mathcal{P}$). Note that αg has source $g^{-1}(x)$ and target y. For objects $x, y \in \operatorname{Ob} \mathcal{T}$, we denote the set of morphisms from x to y by $\mathcal{T}(x, y)$, and in particular, we let $G_x = \mathcal{T}(x, x)$. Transporter categories are examples of *finite EI categories*, i.e, small categories with finitely many morphisms such that every endomorphism is an isomorphism. Representations of these structures have been studied in [9, 10, 21, 22, 24, 25].

Statements in the following proposition are immediate results of the definition, see [25].

Proposition 4.1. *Let $G, \mathcal{P}$ and $\mathcal{T}$ be as above. Then:*

(1) for each object x, $G_x = \{g \in G \mid g(x) = x\}$;
(2) for objects x, y such that $\mathcal{T}(x, y) \neq \emptyset$, both G_x and G_y act freely on $\mathcal{T}(x, y)$;
(3) two objects $x, y \in \operatorname{Ob} \mathcal{T}$ are isomorphic if and only if they are in the same G-orbit.

We focus on the representation type of $\mathcal{T}$. By definition, a *representation* of $\mathcal{T}$ is a covariant functor from $\mathcal{T}$ to k-vec, the category of finite-dimensional vector spaces. Since the category of representations of $\mathcal{T}$ is equivalent to the category of finitely generated $k\mathcal{T}$-modules, where $k\mathcal{T}$ is the *category algebra* of $\mathcal{T}$ [21], we identify in this paper representations of $\mathcal{T}$ and $k\mathcal{T}$-modules. In [25] Xu shows that the category algebra $k\mathcal{T} \cong k\mathcal{P}G$, the skew group algebra of the incidence algebra $k\mathcal{P}$. Clearly, the identity morphisms $\{1_x\}_{x \in \operatorname{Ob} \mathcal{T}}$ form a complete set of primitive orthogonal idempotent of $k\mathcal{P}$ which is closed under the action of G.

We first point out that the representation type of $\mathcal{T}$ is not completely determined by the representation types of kG and $\mathcal{P}$. Actually, the action of G on $\mathcal{P}$ plays an important role, as shown by the following example.

Example 4.2. *Let $\mathcal{P}$ be the following poset and $G = \langle g \rangle$ be a cyclic group of order $p = 3$.*

$$\begin{tikzcd} & x \\ \bullet \arrow[ur] \arrow[r] \arrow[dr] & y \\ & z \end{tikzcd}$$

If G acts trivially on $\mathcal{P}$, then $k\mathcal{T} \cong k\mathcal{P}[X]/(X^3)$. Using covering theory described in [5] *we can show that it is of infinite representation type.*

On the other hand, there is another action such that $g(x) = y$ and $g(y) = z$. In this case the category algebra of the skeletal subcategory $\mathcal{E}$ of $\mathcal{T}$ is isomorphic to the path algebra of the following quiver with relation $\delta^3 = 0$:

$$\mathcal{E}: \quad \begin{tikzcd} x \arrow[loop left, "\delta"] \arrow[r] & y \end{tikzcd} .$$

Since this path algebra is of finite representation type, so is $k\mathcal{T}$.

From this example we find that the finite representation types of G and $\mathcal{P}$ do not guarantee the finite representation type of $\mathcal{T}$. Conversely, suppose that $\mathcal{T}$ is of finite representation type. Although by Corollary 2.2, $\mathcal{P}$ must be of finite representation type as well (finite connected posets of finite representation type have been classified by Loupias in [13]), G may be of infinite representation type, as described in the following example.

Example 4.3. *Let $\mathcal{P}$ be the poset of 4 incomparable elements and G be the symmetric group over 4 letters permuting these 4 incomparable elements. Let k be an algebraically closed field of characteristic 2. Then kG is of infinite representation type. However, the four objects in the transporter category $\mathcal{T}$ are all isomorphic and the automorphism group of each object is a symmetric group H over 3 letters. Therefore, the category algebra $k\mathcal{T}$ is Morita equivalent to kH, which is of finite representation type.*

Now we can show that $\mathcal{T}$ is of finite representation type only for very few cases when $p \neq 2, 3$.

Proposition 4.4. *Let G be a finite group acting on a connected finite poset $\mathcal{P}$ and suppose that $p \neq 2, 3$. Let S be a Sylow p-subgroup of G. If the transporter category $\mathcal{T} = G \propto \mathcal{P}$ is of finite representation type, then one of the following conditions must be true:*

1. *$|G|$ is invertible in k and $\mathcal{P}$ is of finite representation type;*
2. *$\mathcal{P}$ has only one element and G is of finite representation type;*
3. *if $S \neq 1$, then the skeletal category $\mathcal{E}$ of the transporter category $\mathcal{S} = S \propto \mathcal{P}$ is a poset of finite representation type.*

Proof. Suppose that $\mathcal{T}$ is of finite representation type, or equivalently, $k\mathcal{T}$ is of finite representation type. If $|G|$ is invertible, we immediately get (1). If $\mathcal{P}$ has only one element, (2) must be true. Therefore, assume that $|G|$ is not invertible and $\mathcal{P}$ is nontrivial. Thus $S \neq 1$ and $\mathcal{E}$ is a connected skeletal finite EI category with more than one objects. By Corollary 2.2, $\mathcal{S}$ and hence $\mathcal{E}$ are of finite representation type.

Note that $k\mathcal{P}$ is not a local algebra. By Proposition 3.4, S must act freely on $\mathcal{P}$, so $\mathcal{S}(x, x) = 1$ for every $x \in \mathcal{S}$. Consequently, $\mathcal{E}(x, x) = 1$ for every $x \in \operatorname{Ob} \mathcal{E}$, and $\mathcal{E}$ is an acyclic category, i.e., a skeletal finite EI category such that the automorphism group of every object is trivial. Since $\mathcal{E}$ is of finite representation type, we conclude that $|\mathcal{E}(x, y)| \leqslant 1$ for all $x, y \in \operatorname{Ob} \mathcal{E}$. Otherwise, we can find some $x, y \in \operatorname{Ob} \mathcal{E}$ such that there are at least two morphisms from x to y. Considering the full subcategory of $\mathcal{E}$ formed by x and y we get a Kronecker quiver, which is of infinite representation type. This is impossible.

We have shown that $\mathcal{E}$ is a skeletal finite EI category such that the automorphism groups of all objects are trivial and the number of morphisms between any two

distinct objects cannot exceed 1. It is indeed a poset. Clearly, it must be of finite representation type. □

Therefore, for $p \neq 2, 3$ and nontrivial $\mathcal{P}$, if $\mathcal{T}$ is of finite representation type, then all Sylow p-subgroups of G must act freely on the poset $\mathcal{P}$. In the rest of this paper we will show that third case cannot happen. That is, if a nontrivial Sylow p-subgroup $S \leqslant G$ acts freely on the poset, then $\mathcal{T}$ must be of infinite representation type.

First we describe a direct corollary of this proposition.

Corollary 4.5. *Let G be a finite group having a Sylow p-subgroup acting nontrivially on a connected finite poset $\mathcal{P}$. If $p \geqslant 5$ and $\mathcal{P}$ is a directed tree, then the transporter category $\mathcal{T} = G \propto \mathcal{P}$ is of infinite representation type.*

Proof. Without loss of generality we can assume that G is a p-group. Since the action of G is nontrivial, $\mathcal{P}$ must be nontrivial as well. Therefore, if $\mathcal{T}$ is of finite representation type, G must act freely on $\mathcal{P}$. Suppose that $\mathcal{P}$ has n elements, $n \geqslant 2$. Then it has $n-1$ arrows. But G acts freely both on the set of elements and on the set of arrows. Therefore, $|G|$ divides both n and $n-1$. This forces $|G| = 1$, which is impossible. □

The following lemma is useful to determine the representation types of $\mathcal{T}$ for some cases.

Lemma 4.6. *Let G be a finite group acting freely on a connected finite poset $\mathcal{P}$. If there exist distinct elements $x, y, z \in \mathcal{P}$ such that $x < y$ and $x < z$ (or $x > y$ and $x > z$) and y, z are in the same G-orbit, then $\mathcal{T} = G \propto \mathcal{P}$ is of infinite representation type.*

Proof. Note that G is nontrivial since one of its orbits contains at least two elements. Without loss of generality we assume that there are elements $x, y, z \in \mathcal{P}$ such that $x < y$ and $x < z$ where y and z are in the same G-orbit. Then we can find some $1_G \neq g \in G$ with $g(z) = y$. Let α and β be the morphisms $x < y$ and $x < z$ in $\mathcal{P}$ respectively.

Consider the skeletal category $\mathcal{E}$ of $\mathcal{T}$. Since G acts freely on $\mathcal{P}$, $G_u = \mathcal{E}(u, u) = 1$ for every $u \in \operatorname{Ob} \mathcal{P}$. Note that elements in $\mathcal{P}$ lie in disjoint G-orbits, and the objects of $\mathcal{E}$ form a set of representatives of these G-orbits. Clearly, we can let $x, y \in \operatorname{Ob} \mathcal{E}$ since they are in distinct G-orbits.

We claim that there are more than one morphisms from x to y in $\mathcal{E}$. If this is true, then $\mathcal{E}$ must be of infinite representation type since $\mathcal{E}(x, x) \cong 1 \cong \mathcal{E}(y, y)$, and the conclusion follows.

Clearly, $\alpha 1_G$ is a morphism in $\mathcal{E}(x, y)$. Another morphism in this set is $g(\beta)g$. Indeed, let $1_y 1_G$ and $1_x 1_G$ be the identity morphism of y and x in $\mathcal{S}$, we check:

$$1_y 1_G \cdot g(\beta)g = 1_y g(\beta)g = g(\beta)g$$

since $g(\beta)$ has target y; and

$$g(\beta)g \cdot 1_x 1_G = g(\beta 1_x)g = g(\beta)g$$

since β has source x. Thus there are indeed at least two morphisms $\alpha 1_G$ and $g(\beta)g$ in $\mathcal{E}(x, y)$. □

Example 4.7. *Let $\mathcal{P}$ be the following poset and $G = \langle g \rangle$ be a cyclic group of order $p = 3$. The action of G on $\mathcal{P}$ is defined by $g(n) = (n+2) \mod 6$, $1 \leqslant n \leqslant 6$.*

$$\begin{tikzcd}
& & 1 \arrow[dl] \arrow[ddl] \arrow[dddl] \\
& 2 & \\
3 \arrow[ur] \arrow[r] \arrow[dr] & 4 & 5 \arrow[ul] \arrow[l] \arrow[dl] \\
& 6 &
\end{tikzcd}$$

Since G acts freely on $\mathcal{P}$, we get $G_n = 1$ for $1 \leqslant n \leqslant 6$. But for every element in this poset we can find three arrows starting or ending at elements in the same G-orbit. Therefore, the transporter category $G \propto \mathcal{P}$ is of infinite representation type.

The next lemma is used in the proof of the main result of this section.

Lemma 4.8. *Let $\mathcal{P}$ be a connected finite poset and G be a nontrivial finite group acting on it freely. Then there exists a non-oriented cycle $\mathcal{C}$ in $\mathcal{P}$ such that $\mathcal{C}$ is closed under the action of some $1_G \neq g \in G$, and every element in $\mathcal{C}$ (viewed as a poset) is either maximal or minimal.*

Proof. For every element $x \in \mathcal{P}$ we let Gx be the G-orbit where x lies. Since $\mathcal{P}$ is connected, for every $x \in \mathcal{P}$ and each $x \neq y \in Gx$ we can find a non-oriented path connecting x and y. Let γ be a non-oriented path satisfying the following conditions: the two endpoints x and y of γ are distinct and lie in the same G-orbit; γ has the minimal length. This γ exists (although might not be unique) since $\mathcal{P}$ is connected.

Note that the length of γ is at least 2 since its two endpoints are in the same G-orbit and they are incomparable in $\mathcal{P}$. Moreover, distinct elements in γ lie in different G-orbits except the two endpoints. Indeed, if there are two vertices u and v in γ lying in the same G-orbit and $\{u, v\} \neq \{x, y\}$, then we can take a proper segment of γ with endpoints u and v satisfying the required conditions. This contradicts the assumption that γ is shortest among all paths satisfying the required conditions.

By the construction of γ, we can find a unique $1_G \neq g \in G$ such that $y = g(x)$. Note that $n = |g| > 1$. For $0 \leqslant i \leqslant n-1$, let $g^i(\gamma)$ be the corresponding path (which is isomorphic to γ as posets) starting at $g^i(x)$ and ending at $g^i(y)$. Since $g^i(y) = g^{i+1}(x)$, we can glue these paths to get a loop $\tilde{\mathcal{C}}$. Since G acts freely on $\mathcal{P}$, and elements in each $g^i(\gamma)$ (except the two endpoints) are in distinct G-orbits, we deduce that $\tilde{\mathcal{C}}$ has no self-intersection. Therefore, it is a non-oriented cycle, and is closed under the action of g. Now we pick up all maximal elements and minimal elements in $\tilde{\mathcal{C}}$ and get another non-oriented cycle $\mathcal{C}$, which is still closed under the action of g. This cycle $\mathcal{C}$ is what we want. $\square$

The following example explains our construction.

Example 4.9. *Let $\mathcal{P}, G$ and p be as in Example 4.7. Then $\gamma : 1 \to 2 \leftarrow 3$ is a path connecting 1 and 3. The cycle $\mathcal{C}$ is as below:*

$$\begin{tikzcd}
6 & 5 \arrow[l] \arrow[r] & 4 \\
1 \arrow[u] \arrow[r] & 2 & 3 \arrow[l] \arrow[u]
\end{tikzcd}$$

It is closed under the action of G.

Now we can show that the third case in Proposition 4.4 cannot happen.

Proposition 4.10. *Let G be a nontrivial finite group acting freely on a connected finite poset $\mathcal{P}$. Then the transporter category $\mathcal{T} = G \propto \mathcal{P}$ is of infinite representation type.*

Proof. We have two cases.

Case I: There are distinct $x, y, z \in \mathcal{P}$ such that either $x < y$ and $x < z$ or $x > y$ and $x > z$, and y and z are in the same G-orbit. Then by Lemma 4.6, $\mathcal{T}$ is of infinite representation type.

Case II: The situation described in Case I does not happen. By Lemma 4.8, there are non-oriented cycles $\mathcal{C}$ such that every element in $\mathcal{C}$ is either maximal or minimal, and $\mathcal{C}$ is closed under the action of some $1_G \neq g \in G$. Take such a cycle with minimal length and still denote it by $\mathcal{C}$. We claim that $\mathcal{C}$ is a full subcategory of $\mathcal{P}$. To establish this claim, it suffices to show that two incomparable elements in $\mathcal{C}$ are still incomparable in $\mathcal{P}$.

Let x and y be two incomparable elements in $\mathcal{C}$ and suppose that $x < y$ in $\mathcal{P}$. Then x and y must be in distinct G-orbits, and hence in distinct H-orbits, where $H = \langle g \rangle$. Moreover, y cannot be adjacent to some element $x' \in Hx$ in $\mathcal{C}$. Otherwise, we have two possible choices as follows,

$$\begin{tikzcd}
x \arrow[r] \arrow[dr] & y' \\
x' \arrow[r] & y
\end{tikzcd}, \qquad
\begin{tikzcd}
x \arrow[dr] & y' \arrow[l] \\
x' & y \arrow[l]
\end{tikzcd}$$

where y and y' are in the same H-orbit. Note that $x \neq x'$ and $y \neq y'$ since x and y have been assumed to be incomparable in $\mathcal{C}$. But in the first choice we get the situation described Case I, and in the second situation y is neither maximal nor minimal in $\mathcal{C}$. Therefore, both situations cannot happen.

Now let γ be a shortest path in $\mathcal{C}$ connecting x and some element in $x \neq z \in Hx$. Then vertices of γ, except x and z, are contained in distinct H-orbits. Consider the following picture:

$$\begin{tikzcd}
\gamma : x \arrow[r, dash] \arrow[drrr] & x_1 \arrow[r, dash] & \ldots \arrow[r, dash] & y' \arrow[r, dash] & \ldots \arrow[r, dash] & z \\
\gamma' : x' \arrow[r, dash] & x'_1 \arrow[r, dash] & \ldots \arrow[r, dash] & y \arrow[r, dash] & \ldots \arrow[r, dash] & z'
\end{tikzcd},$$

where γ and γ' coincide if y is contained in γ. Note that y is distinct from x, x', z, z' since x, x', z, z' are all in the same H-orbit. Moreover, y is not adjacent to x, x', z, z'

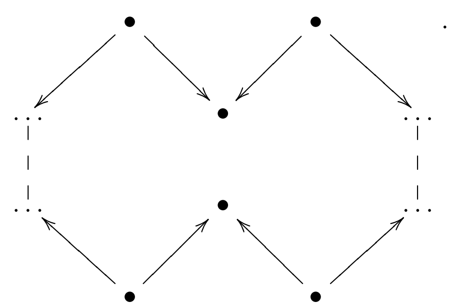
by the reasoning in the previous paragraph. Therefore, we get a non-oriented path:

$$\theta : x \longrightarrow y \longrightarrow \ldots \longrightarrow z' \, .$$

Clearly, the length $l(\theta) < l(\gamma)$.

Now as we did in the proof of Lemma 4.8, by using θ and the action of H on it we obtain a non-oriented cycle $\tilde{\mathcal{D}}$. Picking up maximal and minimal elements in $\tilde{\mathcal{D}}$ (viewed as a poset) we obtain another cycle $\mathcal{D}$ satisfying that it is closed under the action of H and every element in $\mathcal{D}$ (viewed as a poset) is either maximal or minimal. But $\mathcal{D}$ is clearly shorter than $\mathcal{C}$. This contradicts our choice of $\mathcal{C}$. Therefore, x and y are incomparable in $\mathcal{P}$ as well, and our claim is proved. That is, $\mathcal{C}$ is a full subcategory of $\mathcal{P}$.

Since both the path γ and the group H is nontrivial, $\mathcal{C}$ has at least 4 objects. Moreover, every element in $\mathcal{C}$ is either maximal or minimal. Therefore, the structure of $\mathcal{C}$ is as below:



It is of infinite representation type either by Loupias's classification (viewing $\mathcal{C}$ as a poset) [13] or by Gabriel's theorem (viewing $\mathcal{C}$ as a quiver). Therefore, $\mathcal{P}$ is also of infinite representation type. By of Corollary 2.2, $\mathcal{T} = G \propto \mathcal{P}$ is of infinite representation type as well. $\square$

This proposition is true for all characteristics and all finite groups. Note that in Example 4.9 the cycle $\mathcal{C}$ is not a full subcategory of $\mathcal{P}$ since the situation in Case I indeed happens. Moreover, the condition that $\mathcal{P}$ is connected is required. Consider the following example:

Example 4.11. *Let $\mathcal{P}$ be the following poset and $G = \langle g \rangle$ be a cyclic group of order $p = 2$, where $g(i) = i'$, $i = 1, 2$. Then G acts freely on $\mathcal{P}$. But the skeletal category of $\mathcal{T} = G \propto \mathcal{P}$ is isomorphic to a component of $\mathcal{P}$ (viewed as a category), so is of finite representation type.*

$$1 \longrightarrow 2 \qquad 1' \longrightarrow 2' \, .$$

We are ready to classify the representation types of transporter categories for $p \geqslant 5$.

Theorem 4.12. *Let G be a finite group acting on a connected finite poset $\mathcal{P}$ and suppose that $p \neq 2, 3$. Then the transporter category $\mathcal{T} = G \propto \mathcal{P}$ is of finite representation type if and only if one of the following conditions is true:*

(1) $|G|$ is invertible in k and $\mathcal{P}$ is of finite representation type;
(2) $\mathcal{P}$ has only one element and G is of finite representation type.

Proof. It suffices to show that the third case in Proposition 4.4 cannot happen. Let $S \neq 1$ be a Sylow p-subgroup of G and suppose that S acts freely on $\mathcal{P}$. Then $\mathcal{S} = S \propto \mathcal{P}$ is of infinite representation type by the previous proposition, so is $\mathcal{T}$ by Corollary 2.2. □

The conclusion is not true for $p = 2$ or 3, see Example 4.2. For $p = 2$ or 3, in practice we can construct the covering of the skeletal category of $\mathcal{T}$ and use it to examine the representation type. For details, see [5].

5. Koszul Properties of Skew Group Algebras

In this section we study the Koszul properties of skew group algebras. Let $\Lambda = \bigoplus_{i \geqslant 0} \Lambda_i$ be a locally finite graded k-algebra generated in degrees 0 and 1, i.e., $\Lambda_i \Lambda_j = \Lambda_{i+j}$ and $\dim_k \Lambda_i < \infty$ for $i, j \geqslant 0$. Let G be a finite group of grade-preserving algebra automorphisms. That is, for every $i \geqslant 0$ and $g \in G$, $g(\Lambda_i) = \Lambda_i$. Then the skew group algebra ΛG is still a locally finite graded algebra generated in degrees 0 and 1 with $(\Lambda G)_0 = \Lambda_0 \otimes kG$. Since $(\Lambda G)_0$ in general is not semisimple, the classical Koszul theory describe in [3, 17] cannot apply, and we have to rely on generalized Koszul theories not requiring the semisimple property of $(\Lambda G)_0$, for example [8, 11, 12, 14, 23]. In this paper we use the generalized Koszul theory developed in [11, 12].

First we describe some definitions and results of this theory. For proofs and other details, please refer to [11, 12]. Let $A = \bigoplus_{i \geqslant 0} A_i$ be a locally finite graded algebra generated in degrees 0 and 1, and M be a locally finite graded A-module generated in degree 0.

Definition 5.1. *We call M a generalized Koszul module if it has a linear projective resolution*

$$\ldots \longrightarrow P^n \longrightarrow \ldots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow M$$

such that each P^i is generated in degree i. The algebra A is said to be a generalized Koszul algebra if $A_0 \cong A/J$ is a generalized Koszul A-module, where $J = \bigoplus_{i \geqslant 1} A_i$.

It is clear that from this definition M is generalized Koszul if and only if for every $i \geqslant 0$, its i-th syzygy $\Omega^i(M)$ is generated in degree i. For a generalized Koszul module M, since $\Omega^{i+1}(M)$ is generated in degree $i+1$, $\Omega^i(M)_i \cong P^i_i$ is a projective A_0-module.

Note that $\mathrm{Ext}^*_A(M, A_0)$ as an $E = \mathrm{Ext}^*_A(A_0, A_0)$-module has a natural grading. If A_0 is semisimple, it is well known that M is a Koszul A-module if and only if $\mathrm{Ext}^*_A(M, A_0)$ as an E-module is generated in degree 0. This is not true for general A_0. However, we have the following result:

Theorem 5.2. *(Theorem 2.16,* [11].*) Let A, M and E as before. Then M is a generalized Koszul A-module if and only if* $\mathrm{Ext}^*_A(M, A_0)$ *as an E-module is generated in degree 0, and $\Omega^i(M)_i$ is a projective A_0-module for every $i \geqslant 0$.*

We call M a *quasi-Koszul* A-module if $\mathrm{Ext}^*_A(M, A_0)$ as an E-module is generated in degrees 0. Correspondingly, A is a *quasi-Koszul* algebra if E as a graded algebra is generated in degrees 0 and 1.

This generalized Koszul theory and the classical theory are closely related by the following result. Let $\mathfrak{r} = \operatorname{rad} A_0$ and $\mathfrak{R} = A\mathfrak{r}A$ be the two-sided graded ideal. Define $\bar{A} = A/\mathfrak{R}$ and $\bar{M} = M/\mathfrak{R}M$. It is clear that $\bar{A}$ is a graded algebra and $\bar{M}$ is a graded $\bar{A}$-module.

Theorem 5.3. *(Theorem 3.12, [12].) Suppose that both M and A are projective A_0-modules. Then M is a generalized Koszul A-module if and only if $\bar{M}$ is a classical $\bar{A}$-module. In particular, A is a generalized Koszul algebra if and only if $\bar{A}$ is a classical Koszul algebra.*

The graded algebra A is said to have the splitting property (S) if every exact sequence $0 \to P \to Q \to R \to 0$ of left (resp., right) A_0-modules splits whenever P and Q are left (resp., right) projective Λ_0-modules. If A_0 is self-injective or is a direct sum of local algebras, the splitting property is satisfied.

The classical Koszul duality can be generalized as follows:

Theorem 5.4. *(Theorem 4.1, [11].) Suppose that A has the splitting property. If A is a generalized Koszul algebra, then $F = \mathrm{Ext}^*_A(-, A_0)$ gives a duality between the category of generalized Koszul A-modules and the category of generalized Koszul E-modules. That is, if M is a generalized Koszul A-module, then $F(M)$ is a generalized Koszul E-module, and $F_E(F(M)) = \mathrm{Ext}^*_E(F(M), E_0) \cong M$ as graded A-modules.*

Now we consider the generalized Koszul properties of skew group algebra ΛG. Firstly, we observe that for a graded ΛG-module M and for every $i \geqslant 0$, M_i is not only a Λ_0-module, but a kG-module.

Lemma 5.5. *Let M be a graded ΛG-module whose restricted module ${}_\Lambda M$ is a generalized Koszul Λ-module, and let*

$$\ldots \longrightarrow P^n \xrightarrow{d_n} \ldots \longrightarrow P^1 \xrightarrow{d_1} P^0 \xrightarrow{d_0} {}_\Lambda M$$

be a minimal graded projective resolution. Then every P^i is a finitely generated ΛG-module.

Proof. Let K^{i+1} be the kernel of d_i for $i \geqslant 0$. Since ${}_\Lambda M$ is generalized Koszul, P^i is generated in degree i. In particular, $P^0_0 \cong M_0$ is a kG-module. The action of $g \in G$ on $\Lambda \otimes_{\Lambda_0} M_0$ is defined by $g(\lambda \otimes v) = g(\lambda) \otimes g(v)$. Therefore, $P^0 \cong \Lambda \otimes_{\Lambda_0} P^0_0$ is a kG-module, and hence a ΛG-module.

The above resolution induces short exact sequences

$$0 \longrightarrow K^1_j \longrightarrow P^0_j \longrightarrow M_j \longrightarrow 0$$

for $j \geqslant 0$. Thus each K^1_j is also a kG-module, in particular so is K^1_1. Therefore, $P^1 \cong \Lambda \otimes_{\Lambda_0} K^1_1$ is a ΛG-module. By induction, every P^i is a ΛG-module.

Note that the category of locally finite Λ-modules is invariant under taking syzygies. Therefore, each P^i is locally finite. Since it is generated in degree i as a Λ-module, it is a finitely generated Λ-module. Clearly, viewed as a ΛG-module, it is also finitely generated. □

We state the following easy fact as a lemma.

Lemma 5.6. *Let P be a locally finite graded ΛG-module. If viewed as an Λ-module it is graded projective and generated in degree i, then $P \otimes_k kG$ is a graded projective ΛG-module generated in degree i.*

Proof. Clearly $P \otimes_k kG$ is a graded Λ-module generated in degree i, on which Λ acts on the left side and G acts diagonally. Since P is isomorphic to a summand of Λ^n for some $n \geqslant 0$, $P \otimes_k kG$ is isomorphic to a summand of $\Lambda^n \otimes_k kG \cong (\Lambda G)^n$, so is a projective ΛG-module. □

We show that Λ and ΛG share the common property of being generalized Koszul algebras. This generalizes a result of Martinez in [16], where both Λ_0 and kG are supposed to be semisimple. [1]

Theorem 5.7. *Let M be a graded ΛG-module. Then the ΛG-module $M \otimes_k kG$ is a generalized Koszul ΛG-module if and only if viewed as a Λ-module M is generalized Koszul. In particular, ΛG is a generalized Koszul algebra over $(\Lambda G)_0 = \Lambda_0 \otimes_k kG$ if and only if Λ is a generalized Koszul algebra over Λ_0.*

Proof. Since M is a generalized Koszul Λ-module, it has a minimal linear projective resolution

$$\ldots \longrightarrow P^n \longrightarrow \ldots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow M \ ,$$

where each P_i is a graded projective Λ-module generated in degree i.

Applying $- \otimes_k kG$, we get an exact sequence

$$\ldots \longrightarrow P^n \otimes_k kG \longrightarrow \ldots \longrightarrow P^0 \otimes_k kG \longrightarrow M \otimes_k kG \longrightarrow 0 \ .$$

By the previous lemmas, each $P^i \otimes_k kG$ is actually a graded projective ΛG-module generated in degree i. Consequently, this sequence gives rise to linear projective resolution of $M \otimes_k kG$. By definition, $M \otimes_k kG$ is a generalized Koszul ΛG-module.

Conversely, suppose that $M \otimes_k kG$ is a generalized Koszul ΛG-module. It has a linear projective resolution

$$\ldots \longrightarrow \tilde{P}^n \longrightarrow \ldots \longrightarrow \tilde{P}^1 \longrightarrow \tilde{P}^0 \longrightarrow M \otimes_k kG \ .$$

Viewed as Λ-modules, we get a resolution

$$\ldots \longrightarrow {}_\Lambda\tilde{P}^n \longrightarrow \ldots \longrightarrow {}_\Lambda\tilde{P}^1 \longrightarrow {}_\Lambda\tilde{P}^0 \longrightarrow {}_\Lambda(M \otimes_k kG) \longrightarrow 0 \ .$$

Clearly, every ${}_\Lambda\tilde{P}^i$ is a projective Λ-module generated in degree i. Moreover, ${}_\Lambda(M \otimes_k kG) \cong ({}_\Lambda M)^{|G|}$. Therefore, ${}_\Lambda M$ has a linear projective resolution, so is a generalized Koszul Λ-module.

In particular, take $M = \Lambda_0$, which is a ΛG-module, we deduce that $\Lambda_0 \otimes_k kG$ is a generalized Koszul ΛG-module if and only if Λ_0 is a generalized Koszul Λ-module. That is, ΛG is a generalized Koszul algebra if and only if so is Λ. $\square$

Our next goal is to show that the extension algebra $\mathrm{Ext}^*_{\Lambda G}((\Lambda G)_0, (\Lambda G)_0)$ is a skew group algebra of $\mathrm{Ext}^*_\Lambda(\Lambda_0, \Lambda_0)$. This result has been proved by Martinez in [16] under the assumption that both Λ_0 and kG are semismple. With a careful check, we find that his proof actually works for the general situation.

Let M and N be two finitely generated ΛG-modules. Then $\mathrm{Hom}_\Lambda(M, N)$ has a natural ΛG-module structure by letting $(g * f)(v) = gf(g^{-1}v)$ for every $g \in G$ and $v \in M$. It is also well known that $\mathrm{Hom}_{\Lambda G}(M, N) = \mathrm{Hom}_\Lambda(M, N)^G$, the set of all Λ-module homomorphisms fixed by the action of every $g \in G$.

For a fixed $g \in G$, we define a functor $F_g : \Lambda\text{-mod} \to \Lambda\text{-mod}$ as follows. For $X \in \Lambda$-mod, $F_g(X) = X^g = X$ as vector spaces. For every $\lambda \in \Lambda$ and $v \in X^g$, the action of λ on $v \in X^g$ is defined by $\lambda * v = g^{-1}(\lambda)v$. For $f \in \mathrm{Hom}_\Lambda(X, Y)$ where $Y \in \Lambda$-mod, $F_g(f) = f^g$, and $f^g(v) = f(v)$ for every $v \in X^g$. To simplify the notation, we identify f^g and f since they are actually the same map.

[1]In a note sent to the author, applying the Koszul complex Witherspoon proved that if Λ is a Koszul algebra and $\Lambda_0 \cong k$, then ΛG is a generalized Koszul over kG. Her proof actually works for the case that Λ_0 is self-injective since under this condition the generalized Koszul complex is defined. For details, see [11].

If X is also a kG-module, then $X \cong X^g$ as Λ-modules. Indeed, we can find a Λ-module isomorphism $\varphi_g : X \to X^g$ by letting $\varphi_g(v) = g^{-1}(v)$. This is clearly a vector space isomorphism. For every $\lambda \in \Lambda$,

$$\varphi_g(\lambda v) = g^{-1}(\lambda v) = g^{-1}(\lambda)g^{-1}(v) = \lambda * \varphi_g(v),$$

so it is also a Λ-module homomorphism.

If both X and Y are kG-modules, applying the functor F_g to a Λ-module homomorphism $f : X \to Y$, we get the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "g*f"] \arrow[d, "\varphi_g"] & Y \arrow[d, "\varphi_g"] \\ X^g \arrow[r, "f^g=f"] & Y^g. \end{tikzcd}$$

Indeed, for every $v \in X$, we have $f(\varphi_g(v)) = f(g^{-1}(v))$, and

$$\varphi_g((g * f)(v)) = \varphi_g(gf(g^{-1}v)) = g^{-1}gf(g^{-1}v) = f(g^{-1}v).$$

Lemma 5.8. *Let M, N and L be finitely generated ΛG-modules. Then there is a natural action of G on $\mathrm{Ext}^n_\Lambda(M, N)$ such that for $x \in \mathrm{Ext}^s_\Lambda(M, N)$, $y \in \mathrm{Ext}^t_\Lambda(N, L)$, and $g \in G$, $g(y \cdot x) = g(y) \cdot g(x)$, where $\cdot$ is the Yoneda product.*

Proof. This the third statement of Lemma 4 in [16]. We copy the proof here for the convenience of the reader. Represent x as the following extension

$$0 \to N \to X_s \to \ldots \to X_1 \to M \to 0$$

and y as the following extension

$$0 \to L \to Y_t \to \ldots \to Y_1 \to N \to 0.$$

For $g \in G$, applying the functor F_g to the first sequence we get

$$0 \to N^g \to X^g_s \to \ldots \to X^g_1 \to M^g \to 0.$$

Since $\varphi_{g^{-1}}$ gives isomorphisms of Λ-modules $N^g \cong N$ and $M^g \cong M$, we let $g(x)$ be the induced extension:

$$\begin{tikzcd} x : 0 \arrow[r] & N^g \arrow[r] \arrow[d, "\varphi_{g^{-1}}"] & X^g_s \arrow[r] \arrow[d, equal] & \ldots \arrow[r] \arrow[d, equal] & X^g_1 \arrow[r] \arrow[d, equal] & M^g \arrow[r] \arrow[d, "\varphi_{g^{-1}}"] & 0 \\ g(x) : 0 \arrow[r] & N \arrow[r] & X^g_s \arrow[r] & \ldots \arrow[r] & X^g_1 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Similarly, we let $g(y)$ be the extension

$$0 \to L \to Y^g_t \to \ldots \to Y^g_1 \to N.$$

Clearly, $g(y \cdot x) = g(y) \cdot g(x)$. For $h \in G$, note that $\varphi_{g^{-1}}\varphi_{h^{-1}} = \varphi_{(gh)^{-1}}$. We have $(hg)(x) = h(g(x))$. This finishes the proof. $\square$

Lemma 5.9. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of ΛG-modules and X be a ΛG-module. Then the long exact sequences*

$$0 \to \mathrm{Hom}_\Lambda(X, L) \to \mathrm{Hom}_\Lambda(X, M) \to \mathrm{Hom}_\Lambda(X, N) \to \mathrm{Ext}^1_\Lambda(X, L) \to \ldots$$

and

$$0 \to \mathrm{Hom}_\Lambda(N, X) \to \mathrm{Hom}_\Lambda(M, X) \to \mathrm{Hom}_\Lambda(L, X) \to \mathrm{Ext}^1_\Lambda(N, X) \to \ldots$$

are exact sequences of kG-modules.

Proof. This is Corollary 5 in [16]. We have shown that every term is a kG-module in the previous lemma, so it is enough to show that every map in the above sequences is G-equivariant. We prove this fact for the first sequence.

Take $x \in \operatorname{Ext}^s_\Lambda(X, L)$ and $f \in \operatorname{Hom}_{kG}(L, M)$. The map f induces a commutative diagram

$$\begin{tikzcd} x : 0 \arrow[r] & L \arrow[r] \arrow[d, "f"] & E_s \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0\ , \\ y : 0 \arrow[r] & M \arrow[r] & Y_s \arrow[r] & \dots \arrow[r] & Y_1 \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

so $\operatorname{Ext}^s_\Lambda(f, M)(x) = y$. Applying the functor F_g and the Λ-module isomorphism φ_g, we get another commutative diagram

$$\begin{tikzcd} g(x) : 0 \arrow[r] & L \arrow[r] \arrow[d, "F_g(f)"] & E^g_s \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & E^g_1 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0\ . \\ g(y) : 0 \arrow[r] & M \arrow[r] & Y^g_s \arrow[r] & \dots \arrow[r] & Y^g_1 \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

Therefore,

$$g \operatorname{Ext}^s_\Lambda(f, M)(x) = g(y) = \operatorname{Ext}^s_\Lambda(F_g(f), M)(gx) = \operatorname{Ext}^s_\Lambda(f, M)(gx)$$

since $F_g(f) = f$.

Now let $\delta : \operatorname{Ext}^s_\Lambda(X, N) \to \operatorname{Ext}^{s+1}_\Lambda(X, L)$ be the connecting map. Applying F_g and φ_g to the given short exact sequence z we get another short exact sequence isomorphic to the original one. Therefore, $g(z) = z$. Then

$$\delta(g(x)) = z \cdot g(x) = g(z) \cdot g(x) = g(z \cdot x) = g\delta(x),$$

so δ is also G-equivariant as well. □

The following lemma is Lemma 8 in [16].

Lemma 5.10. *Let P be a graded finitely generated projective ΛG-module, N be a graded finitely generated ΛG-module, and W be a kG-module. Then the linear map*

$$\theta : \operatorname{Hom}_\Lambda(P, N) \otimes_k W \to \operatorname{Hom}_{\Lambda G}(P \otimes_k kG, N \otimes_k W)$$

defined by $\theta(f \otimes w)(v \otimes g) = (g * f)(v) \otimes gw$ is a natural isomorphism of ΛG-modules.

Proof. Firstly, this map is Λ-linear by checking

$$(g * f)(\lambda v) = gf(g^{-1}(\lambda v)) = gg^{-1}(\lambda)gf(g^{-1}v) = \lambda(g * f)(v)$$

for $\lambda \in \Lambda$ and $v \in P$. We only need to show that it is also G-equivariant.

There is a natural isomorphism of vector spaces:

$$\varphi : \operatorname{Hom}_k(kG, \operatorname{Hom}_\Lambda(P, N \otimes_k W)) \to \operatorname{Hom}_\Lambda(P \otimes_k kG, N \otimes_k W)$$

defined by

$$\varphi(\gamma)(v \otimes g) = \gamma(g)(v), \quad v \in P, g \in G, \gamma \in \operatorname{Hom}_k(kG, \operatorname{Hom}_\Lambda(P, N \otimes_k W)).$$

This map is G-equivariant. Indeed, for every $g, h \in G$ and $v \in P$, we have

$$\begin{aligned}
\varphi(h*\gamma)(v\otimes g) &= \Big((h*\gamma)(g)\Big)(v) \quad \text{by definition}\\
&= \Big(h\cdot\big(\gamma(h^{-1}g)\big)\Big)(v) \quad \text{since } (h*\gamma)(g) = h\cdot\gamma(h^{-1}g)\\
&= h\gamma(h^{-1}g)(h^{-1}v) \quad \text{since } \gamma(h^{-1}g)\in \mathrm{Hom}_\Lambda(P, N\otimes_k W)\\
&= h\varphi(\gamma)(h^{-1}v\otimes h^{-1}g) \quad \text{by definition}\\
&= h\big(\varphi(\gamma)\big)\big(h^{-1}(v\otimes g)\big) \quad G \text{ acts diagonally}\\
&= \big(h*\varphi(\gamma)\big)(v\otimes g).
\end{aligned}$$

The G-equivariant map φ induces an isomorphism

$$\varphi^G : \mathrm{Hom}_k(kG, \mathrm{Hom}_\Lambda(P, N\otimes_k W))^G \to \mathrm{Hom}_\Lambda(P\otimes_k kG, N\otimes_k W)^G$$

which is actually

$$\varphi^G : \mathrm{Hom}_{kG}(kG, \mathrm{Hom}_\Lambda(P, N\otimes_k W)) \to \mathrm{Hom}_{\Lambda G}(P\otimes_k kG, N\otimes_k W).$$

The conclusion follows by observing

$$\mathrm{Hom}_{kG}(kG, \mathrm{Hom}_\Lambda(P, N\otimes_k W)) \cong \mathrm{Hom}_\Lambda(P, N\otimes_k W)$$

as kG-modules. □

The following lemma is similar to Proposition 9 in [16], with a small modification.

Lemma 5.11. *Let M, N be graded ΛG-modules such that M viewed as a Λ-module is a generalized Koszul. Then for every $s \geqslant 0$ we have a natural ΛG-module isomorphism*

$$\tilde{\theta} : \mathrm{Ext}^s_\Lambda(M, N)\otimes_k kG \to \mathrm{Ext}^s_{\Lambda G}(M\otimes_k kG, N\otimes_k kG).$$

Proof. Take a minimal graded projective resolution for ${}_\Lambda M$:

$$\ldots \longrightarrow P^n \longrightarrow \ldots \longrightarrow P^0 \longrightarrow {}_\Lambda M \longrightarrow 0\ .$$

By Lemma 5.5, every P^i is actually a finitely generated ΛG-module. Applying the functor $\mathrm{Hom}_\Lambda(-, N)$ and $-\otimes_k kG$, we get

$$0 \to \mathrm{Hom}_\Lambda(P^0, N)\otimes_k kG \to \ldots \to \mathrm{Hom}_\Lambda(P^s, N)\otimes_k kG \to \ldots .$$

and deduce that the s-th homology is $\mathrm{Ext}^s_\Lambda(M, N)\otimes_k kG$. By the previous lemma, this complex is isomorphic to

$$0 \to \mathrm{Hom}_{\Lambda G}(P^0\otimes_k kG, N\otimes_k kG) \to \ldots \to \mathrm{Hom}_{\Lambda G}(P^s\otimes_k kG, N\otimes_k kG) \to \ldots,$$

by a natural isomorphism θ. Observe that s-th homology of the last sequence is $\mathrm{Ext}^s_{\Lambda G}(M\otimes_k kG, N\otimes_k kG)$. Therefore, θ determines a ΛG-module isomorphism $\tilde{\theta}$ on homologies. □

Now we can prove:

Theorem 5.12. *Let M be a graded ΛG-module and $\Gamma = \mathrm{Ext}^*_\Lambda(M, M)$. If as a Λ-module M it is generalized Koszul, then $\mathrm{Ext}^*_{\Lambda * G}(M\otimes_k kG, M\otimes_k kG)$ is isomorphic to the skew group algebra ΓG.*

Proof. This is Theorem 10 in [16]. Since the proof is essentially the same, we only give a sketch. For details, please refer to [16]. Since

$$\tilde{\theta} : \mathrm{Ext}^s_\Lambda(M, N) \otimes_k kG \to \mathrm{Ext}^s_{\Lambda G}(M \otimes_k kG, N \otimes_k kG)$$

gives a ΛG-module isomorphism, it suffices to show $\tilde{\theta}$ preserves multiplication. That is, for $x \otimes g \in \mathrm{Ext}^s_\Lambda(M, N) \otimes_k kG$ and $y \otimes h \in \mathrm{Ext}^t_\Lambda(M, N) \otimes_k kG$, we want to have

$$\tilde{\theta}((x \otimes g) \cdot (y \otimes h)) = \tilde{\theta}(x \otimes g) \cdot \tilde{\theta}(y \otimes h). \tag{5.1}$$

Note that

$$\tilde{\theta}((x \otimes g) \cdot (y \otimes h)) = \tilde{\theta}(x \cdot g(y) \otimes gh), \tag{5.2}$$

where $\cdot$ is the Yoneda product. Let $*$ be the opposite product.

We can represent x and y as exact sequences starting and ending at M. We also observe that x and y correspond to maps $f_x : \Omega^s(M) \to M$ and $f_y : \Omega^t(M) \to M$, which are unique up to maps factoring through projective modules.

Using the isomorphism θ defined in Lemma 5.10, we find that $\tilde{\theta}(y \otimes h)$ corresponds to an exact sequence induced from a projective resolution of $M \otimes_k kG$ by the map $\theta(f_y \otimes h)$, and $\tilde{\theta}(f_x \otimes g)$ corresponds to an exact sequence induced from a projective resolution of $\Omega^t(M) \otimes_k kG$ by the map $\Omega^t(\theta(f_x \otimes g))$. Therefore,

$$\tilde{\theta}(x \otimes g) \cdot \tilde{\theta}(y \otimes h) = \tilde{\theta}(y \otimes h) * \tilde{\theta}(x \otimes g) \leftrightarrow \theta(f_y \otimes h) \circ \Omega^t(\theta(f_x \otimes g)), \tag{5.3}$$

here $\circ$ is the usual composite of maps.

We check that $\Omega^t(\theta(f_x \otimes g)) = \theta(\Omega^t(f_x) \otimes g)$, and

$$\theta(f_y \otimes h) \circ \Omega^t(\theta(f_x \otimes g)) = \theta(f_y \otimes h) \circ \theta(\Omega^t(f_x) \otimes g) = \theta\Big(\big(g(f_y)\big) \circ \Omega^t(f_x) \otimes gh\Big). \tag{5.4}$$

Since $g(f_y)$ corresponds to $g(y)$, we have

$$\theta\Big(\big(g(f_y)\big) \circ \Omega^t(f_x) \otimes gh\Big) \leftrightarrow \tilde{\theta}(g(y) * x \otimes gh) = \tilde{\theta}(x \cdot g(y) \otimes gh). \tag{5.5}$$

Putting (5.1 - 5.5) together, we conclude that $\tilde{\theta}$ preserves the multiplication. □

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